

# The Covariant Stark Effect

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## Abstract

This paper examines the Stark effect, as a first order perturbation of manifestly covariant hydrogen-like bound states. These bound states are solutions to a relativistic Schrödinger equation with invariant evolution parameter, and represent mass eigenstates whose eigenvalues correspond to the well-known energy spectrum of the nonrelativistic theory. In analogy to the nonrelativistic case, the off-diagonal perturbation leads to a lifting of the degeneracy in the mass spectrum. In the covariant case, not only do the spectral lines split, but they acquire an imaginary part which is linear in the applied electric field, thus revealing induced bound state decay in first order perturbation theory. This imaginary part results from the coupling of the external field to the non-compact boost generator. In order to recover the conventional first order Stark splitting, we must include a scalar potential term. This term may be understood as a fifth gauge potential, which compensates for dependence of gauge transformations on the invariant evolution parameter.

## 1 Introduction

The Stark effect — the splitting of degenerate spectral lines in an electric field — was an important early success for quantum theory, and has remained a classroom staple, providing

the introduction to perturbation theory for degenerate states. Paired with the Zeeman effect, in which an external magnetic field couples to the diagonal (but otherwise degenerate) angular momentum operator, the Stark effect demonstrates that this same degeneracy rescues the first order perturbation from the coupling of the external electric field to the off-diagonal position operator. Although the non-compact position operator cannot be considered a small perturbation in any rigorous sense, and in non-perturbative solutions, the discrete energy spectrum goes over to a continuous resonance spectrum, [1], the first order nonrelativistic splitting is the basis for the treatment of Stark broadening in spectroscopy. Stark broadening is been an important consideration in plasma physics [2] and has become a practical diagnostic tool in surface science [3] and astronomy [4]. The strong electric fields required to observe the effect (Johannes Stark's 1913 observation was made with field strengths of  $10^5$  V/cm while typical fields may be two orders of magnitude higher [5]), suggest that a relativistically covariant formulation of the problem may be required, especially as the phenomenon is applied to high precision measurement.

In this paper, we discuss the Stark effect as a first order perturbation to a solution of the two body bound state problem in relativistically covariant quantum mechanics. This formulation of the problem is based on Stueckelberg's off-shell kinematics with invariant evolution parameter [6], generalized to the many particle case by Horwitz and Piron [7] (see also [8]). The relaxation of the mass-shell constraint for particle kinematics is required to achieve an action-at-a-distance framework with scalar potential. In this framework, Arshansky and Horwitz [9] obtained exact solutions for relativistic generalizations of the classical central force problems. These wavefunctions form an induced representation of the Lorentz group [10], and are degenerate in the new quantum numbers associated with the enlarged symmetry. Moreover, dipole radiation, emitted in transitions among these bound states, obeys selection rules which are formally identical to those of the nonrelativistic problem but with covariant interpretation [11]. The bound state solutions for the Coulomb problem represent mass eigenstates whose eigenvalues correspond to the well-known energy spectrum of the nonrelativistic theory [9].

The covariant Zeeman effect has been previously obtained [12] and the covariance of the approach permits the application of machinery developed there to the Stark effect. The construction of the action for the induced representation requires care, especially the coupling

to the vector field in a manner which preserves both Lorentz and local gauge invariance. In the case of constant external electromagnetic field, the first order interaction term becomes a scalar contraction of the field strength tensor with the Lorentz generators. The Zeeman effect is then recovered as a magnetic-like field coupled to the rotation generators, and the Stark effect is obtained as an electric-like field coupled to the boost generators. Since the non-compact boost generators have complex eigenvalues, the relativistic bound states decay even at first order. To recover the usual Stark splitting, we must include an external scalar potential involving a coupling to the spacetime position four-vector. This ‘fifth potential’ has a natural interpretation in the pre-Maxwell electromagnetic theory [13], where it plays the role of a gauge field compensating for transformations which depend on the invariant evolution parameter. In the pre-Maxwell theory, the photon kinematics are also off-shell, however the measurement process picks out the zero-mass eigenstate as an equilibrium state [14]. Under this interpretation of the Stark effect, the off-shell photon becomes a necessary corollary to the parameterized quantum mechanics formalism.

The Stueckelberg equation for the two body problem,

$$i\partial_\tau\psi(x_1, x_2, \tau) = K\psi(x_1, x_2, \tau) = \left[ \frac{p_{1\mu}p_1^\mu}{2M_1} + \frac{p_{2\mu}p_2^\mu}{2M_2} + V(x_1, x_2) \right] \psi(x_1, x_2, \tau) \quad (1)$$

is Poincaré invariant and quadratic in the four momenta. The nonrelativistic central force problems may be generalized to covariant form [9] through the replacement

$$r = \sqrt{(\mathbf{r}_1 - \mathbf{r}_2)^2} \quad \longrightarrow \quad \rho = \sqrt{(\mathbf{r}_1 - \mathbf{r}_2)^2 - (t_1 - t_2)^2} \quad (2)$$

in the argument of the usual potentials. Since  $t_1 \rightarrow t_2$  in the Galilean limit, the original nonrelativistic problem is recovered in this limit.

One may separate variables of the center of mass motion and relative motion in the same way as in the nonrelativistic theory,

$$K = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V(\rho), \quad (3)$$

where

$$\begin{aligned} P^\mu &= p_1^\mu + p_2^\mu & M &= M_1 + M_2 \\ p^\mu &= (M_2 p_1^\mu - M_1 p_2^\mu)/M & m &= M_1 M_2 / M. \end{aligned} \quad (4)$$

The reduced motion is then described by the relative Hamiltonian

$$K_{rel} = \frac{p^\mu p_\mu}{2m} + V(\rho) \quad . \quad (5)$$

In order to obtain the correct nonrelativistic limit for the spectrum in the Coulomb problem, one must choose an arbitrary spacelike unit vector  $n_\mu$  ( $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \Rightarrow n^2 = +1$ ) and restrict the spacetime support of the eigenfunctions to a Restricted Minkowski Space (RMS) corresponding to the condition

$$(x_\perp)^2 = [x - (x \cdot n)n]^2 \geq 0, \quad (6)$$

where  $x \equiv x^\mu$  is the *relative* coordinate  $x_1^\mu - x_2^\mu$ , and  $x^2 = x^\mu x_\mu$ . The RMS is transitive and invariant under the  $O(2,1)$  subgroup of  $O(3,1)$  leaving  $n_\mu$  invariant and translations along  $n_\mu$ . The choice of  $n_\mu$  along the  $z$ -axis leads to the parameterization

$$\begin{aligned} y^0 &= \rho \sinh \beta \sin \theta & y^1 &= \rho \cosh \beta \sin \theta \cos \phi \\ y^2 &= \rho \cosh \beta \sin \theta \sin \phi & y^3 &= \rho \cos \theta \end{aligned} \quad (7)$$

for which

$$(y^1)^2 + (y^2)^2 - (y^0)^2 \geq 0. \quad (8)$$

The eigenfunctions of  $K_{rel}$  form irreducible representations of  $SU(1,1)$  — in the double covering of  $O(2,1)$  — parameterized by the spacelike vector  $n_\mu$  stabilized by the particular  $O(2,1)$  [9, 10].

An induced representation of  $SL(2, \mathbb{C})$  was constructed [10], by applying the Lorentz group to the RMS coordinates  $x^\mu$  and the frame orientation  $n_\mu$ , and studying the action on these wavefunctions. A set of wavefunctions with support on  $(n, x)$  where

$$x \in \text{RMS}(n_\mu) = \left\{ x \mid [x - (x \cdot n)n]^2 \geq 0 \right\} \quad (9)$$

may be regarded as functions of the chosen  $n_\mu$  and the coordinates of a standard frame  $y \in \text{RMS}(\hat{n}_\mu)$ , since the Lorentz transformation  $\mathcal{L}$  which performs the mapping  $\hat{n} = \mathcal{L}(n)$  has the property that

$$x \in \text{RMS}(n_\mu) \quad \text{and} \quad y = \mathcal{L}(n) x \quad \implies \quad y \in \text{RMS}(\hat{n}_\mu). \quad (10)$$

For the choice  $\dot{n} = (0, 0, 0, 1)$ , the parameterization (7) may be used for  $y^\mu$ , and the effect on the wavefunctions of a Lorentz transformation  $\Lambda$ , may be seen from the composition

$$\begin{aligned} x \in \text{RMS}(n_\mu) &\xrightarrow{\Lambda} x' \in \text{RMS}(n'_\mu) \\ \uparrow \mathcal{L}(n)^T &\qquad \qquad \downarrow \mathcal{L}(\Lambda n) \\ y \in \text{RMS}(\overset{\circ}{n}_\mu) &\qquad y' \in \text{RMS}(\overset{\circ}{n}'_\mu) \end{aligned} \quad (11)$$

to be

$$\psi_n(y) \rightarrow \psi_n^\Lambda(y) = \psi_{\Lambda^{-1}n}(D^{-1}(\Lambda^{-1}, n) y) \quad (12)$$

where  $\Lambda$  acts directly on  $n_\mu$ . The representations are moved on an orbit generated by this spacelike vector, and the Lorentz transformations act on  $y^\mu$  through the  $O(2,1)$  little group, represented by  $D^{-1}(\Lambda, n)$ , with the property

$$D^{-1}(\Lambda, n) \dot{n} = \mathcal{L}(\Lambda n) \Lambda \mathcal{L}^T(n) \dot{n} \equiv \dot{n}. \quad (13)$$

Expressing the matrix Lorentz generators as

$$(\mathcal{M}^{\sigma\lambda})^{\mu\nu} = g^{\sigma\mu} g^{\lambda\nu} - g^{\sigma\nu} g^{\lambda\mu}, \quad (14)$$

the matrix  $\mathcal{L}^T(n)$  was chosen in [10] to be

$$\mathcal{L}^T(n) = e^{\gamma \mathcal{M}^{23}} e^{\omega \mathcal{M}^{31}} e^{\alpha \mathcal{M}^{03}} \quad (15)$$

$$= \begin{pmatrix} \cosh \alpha & 0 & 0 & \sinh \alpha \\ -\sin \omega \sinh \alpha & \cos \omega & 0 & -\sin \omega \cosh \alpha \\ \sin \gamma \cos \omega \sinh \alpha & \sin \gamma \sin \omega & \cos \gamma & \sin \gamma \cos \omega \cosh \alpha \\ \cos \gamma \cos \omega \sinh \alpha & \cos \gamma \sin \omega & -\sin \gamma & \cos \gamma \cos \omega \cosh \alpha \end{pmatrix}, \quad (16)$$

which provides the parameterization of  $n_\mu$  as

$$n_\mu = \begin{pmatrix} \sinh \alpha \\ -\sin \omega \cosh \alpha \\ \sin \gamma \cos \omega \cosh \alpha \\ \cos \gamma \cos \omega \cosh \alpha \end{pmatrix}. \quad (17)$$

The generators  $h_{\alpha\beta}(n)$  of (12) form a representation of the  $O(3,1)$  Lie algebra (through their action on  $y$  and  $n$ ), and the Casimir operators

$$\hat{c}_1 = \frac{1}{2} h_{\alpha\beta}(n) h^{\alpha\beta}(n) \quad \hat{c}_2 = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} h_{\alpha\beta}(n) h_{\gamma\delta}(n) \quad (18)$$

and the operators of the SU(2) subgroup

$$\mathbf{L}^2(n) = \frac{1}{2} h_{ij}(n) h^{ij}(n) \quad L_1(n) = h^{23}(n) = -i \frac{\partial}{\partial \gamma} \quad (19)$$

can be constructed as a commuting set. Moreover, the operator

$$\Lambda = \frac{1}{2} M^{\mu\nu} M_{\mu\nu} \rightarrow \ell(\ell+1) - \frac{3}{4}, \quad (20)$$

where  $M^{\mu\nu} = y^\mu p^\nu - y^\nu p^\mu$ , and the O(2,1) Casimir  $N^2 = (M^{01})^2 + (M^{02})^2 + (M^{12})^2$  commute with this set. The wavefunctions which are eigenfunctions of the set

$$\{\Lambda, N^2, \hat{c}_1, \hat{c}_2, \mathbf{L}^2(n), L_1(n)\} \quad (21)$$

with eigenvalues  $Q = \{\ell(\ell+1) - \frac{3}{4}, n^2 - \frac{1}{4}, c_1, c_2, L(L+1), q\}$  form a representation of SL(2,C). The requirement that the resulting representation be unitary and irreducible (the wavefunctions lie in the principal series), imposes the condition  $c_1 = \hat{n}^2 - 1 - c_2^2/\hat{n}^2$ , where  $\hat{n} = n + 1/2$ .

The wavefunctions in the induced representation have the explicit form [9]

$$\psi_n^Q(y) = R_{n_a \ell}(\rho) \Theta_\ell^n(\theta) \xi^Q(n_\mu, \beta, \phi) \quad (22)$$

where

$$\Theta_\ell^n(\theta) = (1 - \xi^2)^{-\frac{1}{4}} \sqrt{\frac{2\ell+1}{2} \frac{(\ell-n)!}{(\ell+n)!}} P_\ell^n(\xi) \quad (23)$$

$$\xi^Q(n_\mu, \beta, \phi) = \sum_{k=0}^{L-\hat{n}} \mathcal{D}_k^Q(\alpha, \omega, \gamma) \chi_{n+k}^{-n}(\beta, \phi) \quad (24)$$

$$\chi_{n+k}^{-n}(\beta, \phi) = B_{n+k,n}(\beta) \Phi_{n+k}(\phi) \quad (25)$$

$$B_{n+k,n}(\beta) = (1 - \zeta^2)^{\frac{1}{4}} \sqrt{n \frac{(2n+k)!}{k!}} P_{n+k}^{-n}(\zeta) \quad (26)$$

$$\Phi_{n+k}(\phi) = \frac{1}{\sqrt{2\pi}} e^{i(n+k+\frac{1}{2})\phi} \quad (27)$$

$$\mathcal{D}_k^Q(\alpha, \omega, \gamma) = \Xi_{Lk}^{nc_2}(u) P_{q, -M_k}^L(z) e^{-iq\gamma} \quad (28)$$

$$\Xi_{Lk}^{nc_2}(u) = (-1)^k \sqrt{\frac{(2\hat{n}+k-1)!}{(2\hat{n}-1)!k!}} N_L^Q(1-u^2)^{-\frac{\hat{n}-1}{2}} P_{-\frac{ic_2}{\hat{n}}, \hat{n}+k}^L(u) \quad (29)$$

with  $u = \tanh \alpha$ ,  $z = \sin \omega$ ,  $\xi = \cos \theta$ ,  $\zeta = \tanh \beta$ ,  $M_k = \hat{n} + k$  and  $N_L^Q$  a normalization constant. The functions  $P_\ell^n(\xi)$  are standard Legendre polynomials, and  $P_{ab}^L$  is related to the Jacobi polynomials  $P_k^{\alpha\beta}$  through

$$P_{ab}^L(z) = \frac{i^{a-b}}{2^a} \sqrt{\frac{(L-a)!(L+a)!}{(L-b)!(L+b)!}} (1-z)^{\frac{a-b}{2}} (1+z)^{\frac{a+b}{2}} P_{L-a}^{(a-b, a+b)}(z) \quad (30)$$

These wavefunctions are orthogonal with respect to the measure  $d^4y \, d^4n \, \delta(1-n^2)$ , where

$$\begin{aligned} \int d^4y &= \int_0^\infty d\rho \, \rho^3 \int_{-\infty}^\infty d\beta \, \cosh \beta \int_0^\pi d\theta \, \sin^2 \theta \int_0^{2\pi} d\phi \\ &= \int_0^\infty d\rho \, \rho^3 \int_{-1}^1 d\xi \, \sqrt{1-\xi^2} \int_{-1}^1 d\zeta \, (1-\zeta^2)^{-\frac{3}{2}} \int_0^{2\pi} d\phi \end{aligned} \quad (31)$$

$$\begin{aligned} \int d^4n \, \delta(1-n^2) &= \frac{1}{2} \int_{-\infty}^\infty d\alpha \, \cosh^2 \alpha \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\omega \, \cos \omega \int_0^{2\pi} d\gamma \\ &= \frac{1}{2} \int_{-1}^1 \frac{du}{(1-u^2)^2} \int_{-1}^1 dz \int_0^{2\pi} d\gamma \end{aligned} \quad (32)$$

The remaining “radial” function, after the transformation  $\hat{R}(\rho) = \sqrt{\rho} R(\rho)$  must satisfy an equation which is precisely of the form of the nonrelativistic Schrödinger radial equation in three dimensions (and has the same normalization). The states  $\psi_n(y)$  are then eigenstates of the Lorentz invariant  $K_{rel}$ , whose support is on the  $\text{RMS}(n)$ , with the quantum numbers (21), and a principal quantum number  $n_a$ . In particular, the solutions for the problem corresponding to the Coulomb potential [9] yield bound states with a mass spectrum which coincides with the nonrelativistic Schrödinger energy spectrum.

## 2 Phase Space

The Coulomb interaction has support in the RMS of an arbitrary unit vector  $n_\mu$ . However, it was shown in [11] that under dipole emission, the shift in the eigenvalue of  $L_1(n)$  corresponds to a recoil in the orientation of  $n_\mu$  with respect to the polarization of the emitted or absorbed photon. The dependence of the magnetic quantum number  $q$  on the frame orientation is not surprising, since the operator  $L_1(n)$  belongs to the  $\text{SU}(2)$  subgroup of  $\text{SL}(2, \mathbb{C})$ , and acts on  $n_\mu$ , but not on the RMS coordinates (it was shown in [11] that for  $\Lambda$  a rotation about the 1-axis,  $D^{-1}(\Lambda, n) \equiv 1$ ).

In order to consider the coupling to an external electromagnetic field, we construct a classical Lagrangian, in which  $n_\mu$  plays an explicit dynamical role along with the RMS coordinates  $x_\mu$ . We show that the Lorentz generators are conserved quantities for this action, and construct the Hamiltonian, which may be unambiguously quantized and made locally gauge invariant.

We first consider the classical phase space parameterized by  $(n, y)$  and their  $\tau$ -derivatives. From the known transformation properties,

$$n \rightarrow n' = \Lambda n \quad x \rightarrow x' = \Lambda x \quad (33)$$

we find that

$$x' = \Lambda x = \Lambda \left( \mathcal{L}(n)^T y \right) = \left( \mathcal{L}(\Lambda n)^T \mathcal{L}(\Lambda n) \right) \Lambda \mathcal{L}(n)^T y = \mathcal{L}(n')^T y'. \quad (34)$$

so that  $y$  transforms as

$$y \rightarrow y' = D^{-1}(\Lambda, n) y, \quad (35)$$

where  $D^{-1}(\Lambda, n) = \mathcal{L}(\Lambda n) \Lambda \mathcal{L}(n)^T$  belongs to the  $O(2,1)$  which leaves  $\dot{n}$  invariant, i.e.,

$$D^{-1}(\Lambda, n) \dot{n} = \mathcal{L}(\Lambda n) \Lambda \mathcal{L}(n)^T \dot{n} = \dot{n}. \quad (36)$$

The coordinates thus transform as

$$\Lambda : (n, y) \rightarrow (n, y)' = (\Lambda n, D^{-1}(\Lambda, n)y). \quad (37)$$

Since  $\tau$  is a scalar invariant, the *velocity*  $\dot{n} = dn/d\tau$  transforms as a vector,

$$n' = \Lambda n \implies \dot{n}' = \Lambda \dot{n}. \quad (38)$$

However  $\mathcal{L}(n)$  is now  $\tau$ -dependent through  $n_\mu$ , so that

$$y = \mathcal{L}(n(\tau)) x \implies \dot{y} = \mathcal{L}(n)\dot{x} + \dot{\mathcal{L}}(n)x \quad (39)$$

$$x = \mathcal{L}(n(\tau))^T y \implies \dot{x} = \mathcal{L}(n)^T \dot{y} + \dot{\mathcal{L}}(n)^T y. \quad (40)$$

But since  $d\Lambda/d\tau = 0$ , (39) is nevertheless form invariant:

$$\begin{aligned} (\dot{y})' &= \mathcal{L}(n')\dot{x}' + \dot{\mathcal{L}}(n')x' \\ &= \mathcal{L}(\Lambda n)[\Lambda\dot{x}] + \dot{\mathcal{L}}(\Lambda n)[\Lambda x] \\ &= \mathcal{L}(\Lambda n)\Lambda[\mathcal{L}(n)^T \dot{y} + \dot{\mathcal{L}}(n)^T y] + \dot{\mathcal{L}}(\Lambda n)[\Lambda \mathcal{L}(n)]^T y \\ &= [\mathcal{L}(\Lambda n)\Lambda \mathcal{L}(n)^T] \dot{y} + [\mathcal{L}(\Lambda n)\Lambda \dot{\mathcal{L}}(n)^T + \dot{\mathcal{L}}(\Lambda n)\Lambda \mathcal{L}(n)^T] y \\ &= D^{-1}(\Lambda, n)\dot{y} + \dot{D}^{-1}(\Lambda, n) y \\ &= \frac{d}{d\tau}[D^{-1}(\Lambda, n) y]. \end{aligned} \quad (41)$$



In summary, the *phase space* transforms as:

$$\Lambda : \quad \{(n, y); (\dot{n}, \dot{y})\} \longrightarrow \{(\Lambda n, D^{-1}(\Lambda, n)y); (\Lambda \dot{n}, D^{-1}(\Lambda, n)\dot{y} + \dot{D}^{-1}(\Lambda, n)y)\} . \quad (42)$$

To obtain the classical generators of the Lorentz transformation (37), we expand the matrix form of the Lorentz transformations as

$$\Lambda = 1 + \lambda + o(\lambda^2) \quad (43)$$

and write  $\lambda$  as

$$\lambda = \frac{1}{2} \omega_{\alpha\beta} \mathcal{M}^{\alpha\beta} \quad (44)$$

where  $\omega_{\alpha\beta}$ ,  $\alpha, \beta = 0, \dots, 3$  is (infinitesimal) antisymmetric. The matrix generators

$$\mathcal{M}^{\alpha\beta} = \left. \frac{\partial \lambda}{\partial \omega_{\alpha\beta}} \right|_{\omega=0} \quad (45)$$

are those given in (14). According to (43) and (44), (37) becomes

$$\Lambda : (n, y) \rightarrow (n, y)' = (n + \lambda n, \mathcal{L}(n + \lambda n)(1 + \lambda)\mathcal{L}(n)^T y) + o(\omega^2). \quad (46)$$

Representing the classical generators of  $\xi = (n, y) \rightarrow \xi' = (n', y')$  as

$$X_{\alpha\beta} = \sum_{i=1}^8 \left. \frac{\partial \xi^i}{\partial \omega^{\alpha\beta}} \right|_{\omega=0} \frac{\partial}{\partial \xi^i} \quad (47)$$

where

$$\xi^i = \begin{cases} n^\mu & \text{for } i = 1, \dots, 4, \quad \mu = 0, \dots, 3 \\ y^\mu & \text{for } i = 5, \dots, 8, \quad \mu = 0, \dots, 3 \end{cases} \quad (48)$$

we obtain for  $i = 1, \dots, 4$ ,

$$\sum_{i=1}^4 \left. \frac{\partial \xi^i}{\partial \omega^{\alpha\beta}} \right|_{\omega=0} = (\mathcal{M}_{\alpha\beta})^\mu{}_\nu n^\nu \frac{\partial}{\partial n^\mu} = n_\beta \frac{\partial}{\partial n^\alpha} - n_\alpha \frac{\partial}{\partial n^\beta} \quad (49)$$

which was called  $d(\lambda_{\alpha\beta})$  in [10]. Similarly, for  $i = 5, \dots, 8$ ,

$$\begin{aligned} \left. \frac{\partial \xi^i}{\partial \omega^{\alpha\beta}} \right|_{\omega=0} &= \left. \frac{\partial}{\partial \omega^{\alpha\beta}} [\mathcal{L}(n + \lambda n)(1 + \lambda)\mathcal{L}(n)^T y]^i \right|_{\omega=0} \\ &= \mathcal{L}_{\sigma\beta} \mathcal{L}^\rho{}_\alpha (y^\sigma \frac{\partial}{\partial y^\rho} - y^\rho \frac{\partial}{\partial y^\sigma}) - n_\beta \mathcal{L}^\rho{}_\zeta \frac{\partial}{\partial n^\alpha} \mathcal{L}_\sigma{}^\zeta (y^\sigma \frac{\partial}{\partial y^\rho} - y^\rho \frac{\partial}{\partial y^\sigma}) \end{aligned} \quad (50)$$

which was called  $g(\lambda_{\alpha\beta})$  in [10]. We have used the fact that

$$\mathcal{L}(n)\mathcal{L}(n)^T = 1 \implies \left( \frac{\partial}{\partial n^\mu} \mathcal{L}(n) \right) \mathcal{L}(n)^T + \mathcal{L}(n) \frac{\partial}{\partial n^\mu} \mathcal{L}(n)^T = 0. \quad (51)$$

Finally, we obtain for the classical generators

$$X_{\alpha\beta} = \mathcal{L}_{\sigma\beta} \mathcal{L}^\rho{}_\alpha (y^\sigma \frac{\partial}{\partial y^\rho} - y^\rho \frac{\partial}{\partial y^\sigma}) - n_\beta \mathcal{L}^\rho{}_\zeta \frac{\partial}{\partial n^\alpha} \mathcal{L}^\zeta{}_\sigma (y^\sigma \frac{\partial}{\partial y^\rho} - y^\rho \frac{\partial}{\partial y^\sigma}) + n_\beta \frac{\partial}{\partial n^\alpha} - n_\alpha \frac{\partial}{\partial n^\beta} \quad (52)$$

which was called  $ih_n(\lambda_{\alpha\beta})$  in [10], and shown to satisfy the Lie algebra of  $SL(2, \mathbb{C})$ . It is useful to maintain the matrix notation for  $\mathcal{M}_{\alpha\beta}$  so that (52) may be written as

$$\begin{aligned} X_{\alpha\beta} &= [\mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T]^\mu{}_\nu y^\nu \frac{\partial}{\partial y^\mu} - [\mathcal{L}(\mathcal{M}_{\alpha\beta})^\rho{}_\sigma n^\sigma \frac{\partial}{\partial n^\rho} \mathcal{L}^T]^\mu{}_\nu y^\nu \frac{\partial}{\partial y^\mu} - (\mathcal{M}_{\alpha\beta})^\rho{}_\sigma n^\sigma \frac{\partial}{\partial n^\rho} \\ &= -y^T [\mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T] \nabla_{\mathbf{y}} - y^T \mathcal{L}(n) [n^T \mathcal{M}_{\alpha\beta} \nabla_{\mathbf{n}}] \mathcal{L}^T \nabla_{\mathbf{y}} - n^T \mathcal{M}_{\alpha\beta} \nabla_{\mathbf{n}} \end{aligned} \quad (53)$$

where  $(\nabla_{\mathbf{y}})_\mu = \frac{\partial}{\partial y^\mu}$ . By defining the four matrices

$$S_\mu = \mathcal{L} \frac{\partial}{\partial n^\mu} \mathcal{L}^T \quad \mu = 0, \dots, 3 \quad (54)$$

(which by (51) are antisymmetric) equation (53) becomes

$$X_{\alpha\beta} = - \left\{ y^T [\mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T] \nabla_{\mathbf{y}} + n_\mu (\mathcal{M}_{\alpha\beta})^{\mu\nu} [y^T S_\nu \nabla_{\mathbf{y}} + (\nabla_{\mathbf{n}})_\nu] \right\} \quad (55)$$

In the matrix notation of (55), the generators found in [10] have the form

$$d_n(\lambda) = -n_\mu (\mathcal{M}_{\alpha\beta})^{\mu\nu} (\nabla_{\mathbf{n}})_\nu \quad (56)$$

$$g_n(\lambda) = - \left\{ y^T [\mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T] \nabla_{\mathbf{y}} + n_\mu (\mathcal{M}_{\alpha\beta})^{\mu\nu} y^T S_\nu \nabla_{\mathbf{y}} \right\} \quad (57)$$

For the action in  $(n, y)$  coordinates, we choose the simplest Lagrangian containing a kinetic term for  $n_\mu$ , which is

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m r_0^2 \dot{n}^2 - V(n, x), \quad (58)$$

where the scale factor  $r_0$  is required because  $n_\mu$  is a unit vector. Using (40) to expand  $\dot{x}$ , we may write (58) in the form

$$L = \frac{1}{2} m [\dot{y} + \mathcal{L} \dot{\mathcal{L}}^T y]^2 + \frac{1}{2} m r_0^2 \dot{n}^2 - V(n, \mathcal{L}^T y). \quad (59)$$

Notice that when  $\dot{n} = 0$ , the dynamics depend only on  $\dot{y}$  and so the relative coordinate remains within  $RMS(n)$ . By construction, (59) is Lorentz invariant, and so is invariant under the transformations induced by (55). Therefore, applying Noether's theorem and the Euler-Lagrange equation,

$$0 = \delta L = \frac{\partial L}{\partial \xi^i} \delta \xi^i + \frac{\partial L}{\partial \dot{\xi}^i} \delta \dot{\xi}^i = \left[ \frac{\partial L}{\partial \xi^i} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{\xi}^i} \right] \delta \xi^i + \frac{d}{d\tau} \left[ \frac{\partial L}{\partial \dot{\xi}^i} \delta \xi^i \right], \quad (60)$$

for the variation  $\delta\xi^i = \frac{1}{2}\omega^{\alpha\beta}X_{\alpha\beta}\xi^i$ , one obtains the conservation law

$$\frac{d}{d\tau}[\mathbf{p}^\mu X_{\alpha\beta}y_\mu + \pi^\mu X_{\alpha\beta}n_\mu] = 0 \quad (61)$$

where

$$\mathbf{p}_\mu = \frac{\partial \mathcal{L}}{\partial \dot{y}^\mu} \quad \text{and} \quad \pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{n}^\mu}. \quad (62)$$

Using (55) for  $X_{\alpha\beta}$ , (61) becomes,

$$\frac{d}{d\tau}\{y^T \mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T \mathbf{p} + n_\mu (\mathcal{M}_{\alpha\beta})^{\mu\nu} [y^T S_\nu \mathbf{p} + \pi_\nu]\} = 0. \quad (63)$$

If we understand  $\pi_\nu$ , in the Poisson bracket sense, as a derivative with respect to  $n_\mu$ , then the quantum operators  $h_n(\lambda_{\alpha\beta})$  of [10] now appear as classical constants of the motion for the Lagrangian (58).

To obtain the Hamiltonian, we first observe that  $\mathcal{L}$  depends on  $\tau$  only through  $n$ , so

$$\mathcal{L} \dot{\mathcal{L}}^T = \mathcal{L} \left( \dot{n}^\nu \frac{\partial}{\partial n^\nu} \mathcal{L}^T \right) = \dot{n}^\nu S_\nu \quad (64)$$

Applying (62) to (59),

$$\mathbf{p}_\mu = \frac{\partial \mathcal{L}}{\partial \dot{y}^\mu} = m[\dot{y}_\mu + (\mathcal{L} \dot{\mathcal{L}}^T y)_\mu] \Rightarrow \mathbf{p} = m[\dot{y} + \dot{n}^\nu S_\nu y] \quad (65)$$

and

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{n}^\mu} = mr_0^2 \dot{n}_\mu + m[\dot{y} + \dot{n}^\nu S_\nu y]^T \frac{\partial}{\partial \dot{n}^\mu} [\dot{y} + \dot{n}^\nu S_\nu y] = mr_0^2 \dot{n}_\mu - y^T S_\mu \mathbf{p} \quad (66)$$

where we used (65) and the antisymmetry of  $S_\mu$  to obtain (66). Equations (65) and (66) may be inverted to eliminate  $(\dot{n}, \dot{y})$ :

$$\dot{n}_\mu = \frac{1}{mr_0^2} [\pi_\mu + y^T S_\mu \mathbf{p}] \quad (67)$$

and

$$\dot{y} = \frac{1}{m} \mathbf{p} - \dot{n}^\mu S_\mu y = \frac{1}{m} \dot{y} = \frac{1}{m} \mathbf{p} - \dot{n}^\mu S_\mu y = \frac{1}{m} \mathbf{p} - \frac{1}{mr_0^2} [\pi^\mu + y^T S^\mu \mathbf{p}] S_\mu y \quad (68)$$

which may be used to write the Hamiltonian as

$$\begin{aligned} K &= \dot{y} \cdot \mathbf{p} + \dot{n} \cdot \pi - \mathcal{L} \\ &= \frac{\mathbf{p}^2}{2m} + \frac{1}{2mr_0^2} (\pi^\mu + y^T S^\mu \mathbf{p}) (\pi_\mu + y^T S_\mu \mathbf{p}) + V \end{aligned} \quad (69)$$

Since  $S^\mu$  is antisymmetric, we may regard (69) as a quantum Hamiltonian without ordering ambiguity in the operator  $y^T S^\mu \mathbf{p}$ . The Schrödinger equation is then

$$i\partial_\tau \psi = K\psi = \left[ \frac{\mathbf{p}^2}{2m} + \frac{1}{2mr_0^2}(\pi^\mu + y^T S^\mu \mathbf{p})(\pi_\mu + y^T S_\mu \mathbf{p}) + V \right] \psi, \quad (70)$$

where we take as quantum operators

$$\mathbf{p}_\mu = -i \frac{\partial}{\partial y^\mu} \quad \pi_\mu = -i \frac{\partial}{\partial n^\mu} \quad (71)$$

We require that (70) be locally gauge invariant in the coordinate space  $(n, y)$ , that is, under transformations of the form

$$\psi \longrightarrow e^{-ie\Theta(n,y)} \psi; \quad (72)$$

this can be accomplished through the minimal coupling prescription

$$\mathbf{p}_\mu \longrightarrow \mathbf{p}_\mu - e\mathbf{A}_\mu^{(n)} \quad \pi_\mu \longrightarrow \pi_\mu - e\chi_\mu \quad (73)$$

together with the requirement that under gauge transformation

$$\mathbf{A}_\mu^{(n)} \longrightarrow \mathbf{A}_\mu^{(n)} + \frac{\partial}{\partial y^\mu} \Theta \quad \chi_\mu \longrightarrow \chi_\mu + \left( \frac{\partial}{\partial n^\mu} + y^T S_\mu \nabla_{\mathbf{y}} \right) \Theta. \quad (74)$$

Note that  $\mathbf{A}_\mu^{(n)}$  transforms under  $O(3,1)$  as an induced (over  $O(2,1)$ ) representation; it transforms as  $\mathbf{p}_\mu$  under Lorentz transformations (i.e., under the  $O(2,1)$  little group) and so, since the Maxwell equations are Lorentz invariant, it satisfies the Maxwell equation in the  $y^\mu$  variables. Under gauge transformation,

$$(\mathbf{p} - e\mathbf{A}^{(n)'})e^{-ie\Theta}\psi = e^{-ie\Theta}(\mathbf{p} + e\nabla_{\mathbf{y}}\Theta - e\mathbf{A}^{(n)'})\psi = e^{-ie\Theta}(\mathbf{p} - e\mathbf{A}^{(n)})\psi \quad (75)$$

and

$$\begin{aligned} (\pi_\mu + y^T S_\mu \mathbf{p} - e\chi'_\mu)e^{-ie\Theta}\psi &= e^{-ie\Theta}(\pi_\mu + y^T S_\mu \mathbf{p} + e\frac{\partial}{\partial n^\mu}\Theta + ey^T S_\mu \nabla_{\mathbf{n}}\Theta - e\chi'_\mu)\psi \\ &= e^{-ie\Theta}(\pi_\mu + y^T S_\mu \mathbf{p} - e\chi_\mu)\psi, \end{aligned} \quad (76)$$

so that the gauge invariant form of (70) is

$$i\partial_\tau \psi = K\psi = \left[ \frac{1}{2m}(\mathbf{p} - e\mathbf{A}^{(n)})^2 + \frac{1}{2mr_0^2}(\pi^\mu + y^T S^\mu \mathbf{p} - e\chi^\mu)(\pi_\mu + y^T S_\mu \mathbf{p} - e\chi_\mu) + V \right] \psi. \quad (77)$$

Notice the operator

$$D_\mu = \frac{\partial}{\partial n^\mu} + y^T S_\mu \nabla_{\mathbf{y}} = (\nabla_n)_\mu + y^T S_\mu \nabla_{\mathbf{y}} \quad (78)$$

which appears in the second of (74) and in (55). For a function  $f(n, y)$  defined such that its dependence on  $n$  is only through  $\mathcal{L}(n)^T y$  (which is to say that  $f$  is a function of  $x$  alone, even as  $n$  varies in  $\tau$ ), we find that

$$\frac{\partial}{\partial y^\mu} f = \left. \frac{df}{d\xi^\alpha} \right|_{\xi=\mathcal{L}(n)^T y} \frac{\partial}{\partial y^\mu} (\mathcal{L}_\beta^\alpha y^\beta) = \mathcal{L}_\mu^\alpha \left. \frac{df}{d\xi^\alpha} \right|_{\xi=\mathcal{L}(n)^T y} \quad (79)$$

and

$$\frac{\partial}{\partial n^\mu} f = \left. \frac{df}{d\xi^\alpha} \right|_{\xi=\mathcal{L}(n)^T y} \frac{\partial}{\partial n^\mu} (\mathcal{L}_\beta^\alpha y^\beta) \quad (80)$$

so that

$$\begin{aligned} D_\mu f &= \left( \frac{\partial}{\partial n^\mu} + y^T S_\mu \nabla_{\mathbf{y}} \right) f \\ &= \left[ \frac{\partial}{\partial n^\mu} + y_\beta \mathcal{L}^\beta_\gamma \left( \frac{\partial}{\partial n^\mu} \mathcal{L}^{\alpha\gamma} \right) \frac{\partial}{\partial y^\alpha} \right] f \\ &= \left. \frac{df}{d\xi^\sigma} \right|_{\xi=\mathcal{L}(n)^T y} y^\beta \left[ \frac{\partial}{\partial n^\mu} \mathcal{L}_\beta^\sigma + \mathcal{L}_\beta^\gamma \left( \frac{\partial}{\partial n^\mu} \mathcal{L}_\gamma^\alpha \right) \mathcal{L}_\alpha^\sigma \right] \\ &= \left. \frac{df}{d\xi^\sigma} \right|_{\xi=\mathcal{L}(n)^T y} y^\beta \left[ \frac{\partial}{\partial n^\mu} \mathcal{L}_\beta^\sigma + \mathcal{L}_\beta^\gamma (\mathcal{L}^T)^\sigma_\alpha \frac{\partial}{\partial n^\mu} \mathcal{L}_\gamma^\alpha \right] \\ &= \left. \frac{df}{d\xi^\sigma} \right|_{\xi=\mathcal{L}(n)^T y} y^\beta \left[ \frac{\partial}{\partial n^\mu} \mathcal{L}_\beta^\sigma - \mathcal{L}_\beta^\gamma \mathcal{L}_\gamma^\alpha \frac{\partial}{\partial n^\mu} \mathcal{L}_\alpha^\sigma \right] \\ &\equiv 0 \end{aligned} \quad (81)$$

where we have used (51). In fact, it follows from (54) that

$$dx \cdot \nabla_{\mathbf{x}} + dn^\mu D_\mu = dy \cdot \nabla_{\mathbf{y}} + dn \cdot \nabla_{\mathbf{n}} \quad (82)$$

which shows that  $\nabla_{\mathbf{x}}$  and  $D_\mu$  generate the variations induced by  $dx$  and  $dn$ , just as  $\nabla_{\mathbf{y}}$  and  $\nabla_{\mathbf{n}}$  generate the variations induced by  $dy$  and  $dn$ . Thus,  $D_\mu$  acts as a kind of covariant derivative which vanishes on functions of  $x$ . In particular,  $D_\mu$  vanishes on the eigenstates discussed in [9] and [10], in which case the Hamiltonian (69) reduces to the RMS Hamiltonian discussed in [9].

The classical Lagrangian associated with the locally gauge invariant Hamiltonian (69) is

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m r_0^2 \dot{n}^2 + e [\dot{x} \cdot (\mathcal{L}^T A^{(n)}) + \dot{n} \cdot \chi] - V(n, x). \quad (83)$$

In order for  $L$  to be a Lorentz scalar,  $\mathcal{L}^T A^{(n)}$  must transform under the full Lorentz group  $O(3,1)$ . Since  $A^{(n)}$  was introduced as a field which transforms under the  $O(2,1)$  little group,

we have that

$$A^{(n)'} = D^{-1}(\Lambda, n)A^{(n)} = \mathcal{L}(\Lambda n) \Lambda \mathcal{L}^T(n)A^{(n)} . \quad (84)$$

Operating on (84) with  $\mathcal{L}^T(\Lambda n)$  leads to

$$\Lambda \left[ \mathcal{L}^T(n)A^{(n)} \right] = \mathcal{L}^T(\Lambda n)A^{(n)'} = \left[ \mathcal{L}^T(n)A^{(n)} \right]' \quad (85)$$

verifying that the combination  $\mathcal{L}^T A^{(n)}$  transforms as a four vector under  $\Lambda$ .

### 3 Interaction With an External Field

In (73), we introduced the gauge compensation fields,  $A_\mu^{(n)}$  and  $\chi_\mu$ , required to make the Hamiltonian (69) locally gauge invariant. To avoid introducing extra degrees of freedom, we argue that just as  $n$  and  $y$  transform under inequivalent representations of the Lorentz group ( $y$  transforms under the  $O(2,1)$  little group induced by the action of the full  $O(3,1)$ ), so  $A_\mu^{(n)}$  and  $\chi_\mu$  should be seen as inequivalent representations of the usual  $U(1)$  gauge group of electromagnetism. In the full spacelike region, a constant electromagnetic field,  $F^{\mu\nu}$ , can be represented through the vector potential

$$A^\mu(x) = -\frac{1}{2}F^{\mu\nu}x_\nu. \quad (86)$$

We now restrict the support of  $A^\mu$  to  $x \in \text{RMS}(n)$  and express the vector potential as a vector oriented with  $\text{RMS}(\hat{n})$  by writing

$$A_\mu^{(n)}(y) = \mathcal{L}_{\mu\nu}A^\nu(\mathcal{L}^T y) = -\frac{1}{2}\mathcal{L}_{\mu\nu}F^\nu{}_\sigma \mathcal{L}^\sigma{}_\lambda y^\lambda = -\frac{1}{2}(\mathcal{L}F\mathcal{L}^T y)_\mu. \quad (87)$$

For the field  $\chi_\mu$ , we choose (note that  $n$  undergoes Lorentz transform in the same way as  $x$ ),

$$\chi_\mu(n) = b^2 A_\mu(n) = -\frac{b^2}{2} F^\nu{}_\sigma n^\sigma \quad (88)$$

(here  $b$  is another length scale, required since  $A_\mu(x)$  has units of  $\text{length}^{-1}$ , so  $F^\nu{}_\sigma$  must have units of  $\text{length}^{-2}$ , but  $\chi_\mu$  must be without units) and we use (87) and (88) in the Schrödinger equation (77).

$$\begin{aligned} i\partial_\tau \psi &= \left[ \frac{1}{2m}(\mathbf{p} - eA^{(n)})^2 + \frac{1}{2mr_0^2}(\pi^\mu + y^T S^\mu \mathbf{p} - e\chi^\mu)(\pi_\mu + y^T S_\mu \mathbf{p} - e\chi_\mu) + V \right] \psi \\ &= \left[ \frac{1}{2m}\mathbf{p}^2 - \frac{e}{2m}(\mathbf{p} \cdot A^{(n)} + A^{(n)} \cdot \mathbf{p}) + \frac{1}{2mr_0^2}(\pi^\mu + y^T S^\mu \mathbf{p})^2 - \right. \end{aligned}$$

$$\begin{aligned}
& \frac{e}{2mr_0^2}[(\pi^\mu + y^T S^\mu \mathbf{p})\chi_\mu + \chi^\mu(\pi_\mu + y^T S_\mu \mathbf{p})] + V + o(e^2) \Big] \psi \\
& = \left\{ \frac{1}{2m} \mathbf{p}^2 + \frac{1}{2mr_0^2}(\pi^\mu + y^T S^\mu \mathbf{p})^2 + V \right. \\
& \quad \left. - e \left[ \frac{1}{m} \mathbf{A}^{(n)} \cdot \mathbf{p} + \frac{1}{mr_0^2} \chi^\mu(\pi_\mu + y^T S_\mu \mathbf{p}) \right] + o(e^2) \right\} \psi
\end{aligned} \tag{89}$$

where the first three terms of (89) are the unperturbed Hamiltonian  $K_0$ .

The perturbation term to order  $o(e)$ , is

$$\begin{aligned}
& -e \left[ \frac{1}{m} \mathbf{A}^{(n)} \cdot \mathbf{p} + \frac{1}{mr_0^2} \chi^\mu(\pi_\mu + y^T S_\mu \mathbf{p}) \right] \\
& = -e \left[ \frac{1}{m} \mathbf{A}^{(n)T} \mathbf{p} + \frac{1}{mr_0^2} (\chi^T \pi + y^T (S \cdot \chi) \mathbf{p}) \right] \\
& = -\frac{e}{2} \left[ \frac{1}{m} (\mathcal{L} F \mathcal{L}^T y)^T \mathbf{p} + \frac{b^2}{mr_0^2} F^\mu{}_\nu n^\nu (\pi_\mu + y^T S_\mu \mathbf{p}) \right] \\
& = \frac{e}{2m} [y^T \mathcal{L} F \mathcal{L}^T \mathbf{p} + \frac{mb^2}{mr_0^2} n_\nu F^{\nu\mu} (\pi_\mu + y^T S_\mu \mathbf{p})].
\end{aligned} \tag{90}$$

Expanding the electromagnetic field tensor on the basis of four by four antisymmetric tensors given by the Lorentz generators  $\mathcal{M}^{\mu\nu}$ ,

$$F = \frac{1}{2} F_{\mu\nu} \mathcal{M}^{\mu\nu} \implies (F)^{\alpha\beta} = \frac{1}{2} F_{\mu\nu} (\mathcal{M}^{\mu\nu})^{\alpha\beta} = \frac{1}{2} F_{\mu\nu} (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) = F^{\alpha\beta}. \tag{91}$$

Using (91) in (90) we find that the perturbation term to order  $o(e)$  becomes

$$\frac{e}{4m} F_{\alpha\beta} [y^T \mathcal{L} \mathcal{M}^{\alpha\beta} \mathcal{L}^T \mathbf{p} + \frac{b^2}{r_0^2} n_\mu (\mathcal{M}^{\alpha\beta})^{\mu\nu} (\pi_\nu + y^T S_\nu \mathbf{p})] \tag{92}$$

Taking  $b = r_0$ , then we may write the first order perturbation (using (55)) as

$$\frac{e}{4m} F_{\alpha\beta} [y^T \mathcal{L} \mathcal{M}^{\alpha\beta} \mathcal{L}^T \mathbf{p} + n_\mu (\mathcal{M}^{\alpha\beta})^{\mu\nu} (\pi_\nu + y^T S_\nu \mathbf{p})] = \frac{e}{4m} F_{\alpha\beta} X^{\alpha\beta}. \tag{93}$$

The interaction term in (93) was used in [12] to obtain the Zeeman effect. For the magnetic-like field with  $F^{\mu\nu} F_{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2) > 0$ , there exists a frame for which the interaction is purely magnetic. In such a frame, the perturbation becomes

$$\frac{e}{4m} F_{\alpha\beta} X^{\alpha\beta} = \frac{e}{4m} F_{ij} X^{ij} = \frac{e}{4m} \epsilon_{ijk} B^k X^{ij} = \frac{e}{2m} B^k \left[ \frac{1}{2} \epsilon_{ijk} X^{ij} \right] = \frac{e}{2m} B^k h(\lambda_k) \tag{94}$$

where  $h(\lambda_k)$  are the three conserved generators of the  $SU(2)$  rotation subgroup of  $SL(2, \mathbb{C})$  for the phase space  $\{(n, y); (\pi, \mathbf{p})\}$ , that is, the angular momentum operator for the eigenstates

of the induced representation. Notice that in the matrix element for unperturbed eigenstates, the second terms of (90) vanishes, so the relativistic Zeeman effect does not depend upon the values of  $r_0$  or  $b$ .

In [10], the diagonal angular momentum operator is  $L_1(n) = h(\lambda_1) = -i\partial/\partial\gamma$ , and so if we take  $\mathbf{B} = B(1, 0, 0)$  then we find that

$$K_0 \longrightarrow K = K_0 - \frac{eB}{2m}h(\lambda_1) \quad (95)$$

splits the mass levels of the bound states according to

$$K_{\ell n} \longrightarrow K'_{\ell n q} = K_{\ell n} - \frac{eB}{2m}q \quad (96)$$

In going from (95) to (96), we have used the fact that the unperturbed Hamiltonian of (89) reduces to the unperturbed Hamiltonian of [10]. Equation (96) further justifies the conclusion reached in [11] that  $q$  is the magnetic quantum number. As pointed out in [10], the quantum number  $q$  belongs to a representation in the double covering of the Lorentz group, which takes on, in fact, half-integer value, and indicates even multiplicity for the normal Zeeman splittings. Moreover, the manifest covariance of the formalism guarantees that the splitting of the spectrum will be independent of the observer.

## 4 The Stark Effect

For the electric-like field with  $F^{\mu\nu}F_{\mu\nu} < 0$ , we may find a frame in which the interaction is purely electric, leading to the covariant formulation of the Stark effect. In this case, we find from (93) that the first order perturbation is

$$\frac{e}{4m}F_{\alpha\beta}X^{\alpha\beta} = \frac{e}{2m}E^j i h_n(\lambda_{0j}) \quad (97)$$

and the electric field couples to the boost generators, which are off-diagonal, non-compact, and anti-Hermitian [10]. In order to recover the usual Stark level splitting, we propose a second contribution to the perturbation, given by the scalar potential

$$V'(x, n) = -e[-\varepsilon^\mu(x_\mu + r_0 n_\mu)] \ , \quad (98)$$

where  $\varepsilon^\mu$  is a constant four-vector. Together, the perturbation is

$$K' = \frac{e}{2m}E^1 i h_n(\lambda_{01}) + e\varepsilon^1(x_1 + r_0 n_1) \ , \quad (99)$$



where we have taken the fields along the 1-axis.

We first consider separately the contribution from the usual electric field; that is, we take  $\varepsilon^1 = 0$  in (99). The matrix elements for the boost generators follow from directly their algebraic properties [10], and so

$$\begin{aligned} < n_{a'} \ell' n' L' q' c'_2 | i h_n (\lambda_{01}) | n_a \ell n L q c_2 > = \delta_{qq'} \delta_{nn'} \delta_{\ell', \ell} \delta_{n_{a'}, n_a} \delta(c_2 - c'_2) \\ & \times \left[ i C_L \sqrt{L^2 - q^2} \delta_{L', L-1} - i A_L q \delta_{L', L} - i C_{L+1} \sqrt{(L+1)^2 - q^2} \delta_{L', L+1} \right] \end{aligned} \quad (100)$$

where

$$i C_L = -\frac{1}{L} \sqrt{\frac{(L^2 - \hat{n}^2)(L^2 + c_2^2/\hat{n}^2)}{4L^2 - 1}} \quad (101)$$

$$i A_L = \frac{i c_2}{L(L+1)} \quad (102)$$

$$i C_{L+1} = -\frac{1}{L+1} \sqrt{\frac{((L+1)^2 - \hat{n}^2)((L+1)^2 + c_2^2/\hat{n}^2)}{4(L+1)^2 - 1}}. \quad (103)$$

The contribution to the spectrum becomes

$$\begin{aligned} < n_{a'} \ell' n' L' q' c'_2 | K' | n_a \ell n L q c_2 > = \frac{e}{2m} E \delta_{n_{a'}, n_a} \delta_{qq'} \delta_{nn'} \delta_{\ell', \ell} \delta(c_2 - c'_2) \\ & \times \left[ -\frac{\sqrt{L^2 - q^2}}{L} \sqrt{\frac{(L^2 - \hat{n}^2)(L^2 + c_2^2/\hat{n}^2)}{4L^2 - 1}} \delta_{L', L-1} - \frac{i c_2}{L(L+1)} q \delta_{L', L} \right. \\ & \left. + \frac{\sqrt{(L+1)^2 - q^2}}{L+1} \sqrt{\frac{((L+1)^2 - \hat{n}^2)((L+1)^2 + c_2^2/\hat{n}^2)}{4(L+1)^2 - 1}} \delta_{L', L+1} \right]. \end{aligned} \quad (104)$$

We consider the contribution of this term to the ground state, with the quantum numbers

$$n = 0 \quad \ell = 0 \quad n_a = 0 \quad \ell + n_a = 0 \quad L = 1/2, 3/2 \quad q = \pm 1/2 \quad (105)$$

where we recall the even multiplicity for the relativistic ground state. Combining (104) and (105),

$$\begin{aligned} < n_{a'} \ell' n' L' q' c'_2 | K' | n_a \ell n L q c_2 > = \frac{eE}{2m} \delta_{n_{a'}, n_a} \delta_{qq'} \delta_{nn'} \delta_{\ell', \ell} \delta(c_2 - c'_2) \\ & \times \left[ (-q) \left( \frac{4i c_2}{3} \delta_{L, \frac{1}{2}} \delta_{L', \frac{1}{2}} + \frac{4i c_2}{15} \delta_{L, \frac{3}{2}} \delta_{L', \frac{3}{2}} \right) \right. \\ & \left. + \frac{\sqrt{2}}{3} \sqrt{\frac{9}{4} + 4c_2^2} \left( \delta_{L, \frac{3}{2}} \delta_{L', \frac{1}{2}} - \delta_{L, \frac{1}{2}} \delta_{L', \frac{3}{2}} \right) \right] \end{aligned} \quad (106)$$

which after diagonalization, provides the following contribution to the spectrum

$$\Delta K = \pm i \frac{eE}{2m} \left( \frac{6}{15} c_2 \pm \sqrt{\left(\frac{4}{15} c_2\right)^2 + \left(\frac{1}{4} + \frac{4}{9} c_2^2\right)} \right) \xrightarrow{c_2 \rightarrow 0} \pm i \frac{eE}{4m} \quad (107)$$

Since the contribution in (107) is pure imaginary, we see that the usual electric field leads to the decay of the ground state.

We now consider the scalar contribution to (99); that is we consider the case in which  $E^1 = 0$ . The matrix elements for the operators  $x^\mu$  and  $n^\mu$  were computed in [11], and the relevant results are

$$\begin{aligned} \langle n_{a'} \ell' n' L' q' c'_2 | x^1 | n_a \ell n L q c_2 \rangle &= \langle n_{a'} \ell' | \rho | n_a \ell \rangle \frac{q}{L(L+1)} \delta_{qq'} \delta_{nn'} \delta_{L'L} \\ &\times \sum_{i=\pm 1} E_{\ell n}^{(i)} \delta_{\ell' \ell+i} \delta(c_2 - c'_2) \end{aligned} \quad (108)$$

where

$$E_{\ell n}^{(i)} = \begin{cases} (\ell - n + 1) \sqrt{\frac{1}{2\ell+1} \frac{1}{2\ell'+1} \frac{(\ell-n)! (\ell'+n)!}{(\ell+n)! (\ell'-n)!}}, & i = +1 \\ (\ell + n) \sqrt{\frac{1}{2\ell+1} \frac{1}{2\ell'+1} \frac{(\ell-n)! (\ell'+n)!}{(\ell+n)! (\ell'-n)!}}, & i = -1 \end{cases} \quad (109)$$

and

$$\langle n_{a'} \ell' n' L' q' c'_2 | n^1 | n_a \ell n L q c_2 \rangle = \frac{q}{L(L+1)} \delta_{qq'} \delta_{nn'} \delta_{L'L} \delta_{\ell\ell'} \delta_{n_a n'_a} \delta(c_2 - c'_2). \quad (110)$$

Collecting (98), (109), and (110), the perturbative contribution of the scalar term to the spectrum will be,

$$\begin{aligned} \langle n_{a'} \ell' n' L' q' c'_2 | V' | n_a \ell n L q c_2 \rangle &= e\epsilon \frac{q}{L(L+1)} \delta_{qq'} \delta_{nn'} \delta_{L'L} \delta(c_2 - c'_2) \\ &\times \left[ \langle n_{a'} \ell' | \rho | n_a \ell \rangle \sum_{i=\pm 1} E_{\ell n}^{(i)} \delta_{\ell' \ell+i} + r_0 \delta_{\ell\ell'} \delta_{n_a n'_a} \right] \end{aligned} \quad (111)$$

Considering specifically the level splitting in the  $2s - 2p$  system, with the quantum numbers

$$n = 0 \quad L = 1/2 \quad q = \pm 1/2 \quad \ell = 0, 1 \quad n_a = 0, 1 \quad \ell + n_a = 1 \quad (112)$$

we combine (111) and (112) to find

$$\begin{aligned} \langle n_{a'} \ell' n' L' q' c'_2 | V' | n_a \ell n L q c_2 \rangle &= e\epsilon \operatorname{sgn}(q) \delta_{qq'} \delta_{nn'} \delta_{L'L} \delta(c_2 - c'_2) \\ &\times \left[ \frac{2}{3} r_0 \delta_{\ell\ell'} + 2a_0 (\delta_{\ell', \ell-1} + \delta_{\ell', \ell+1}) \right]. \end{aligned} \quad (113)$$

where  $a_0 = \hbar^2/(me^2)$  is the Bohr radius, which enters through the expectation value of  $\rho$  with respect to the radial wavefunctions. After diagonalization, the contribution to the spectrum is

$$\Delta K = \pm e\epsilon \left( \frac{2}{3}r_0 \pm 2a_0 \right) , \quad (114)$$

which may be compared with the standard nonrelativistic result

$$\Delta K_{\text{nonrelativistic}} = \pm eE (3a_0) . \quad (115)$$

The free parameter  $r_0$  appears to be remain for comparison with experiment.

## 5 Interpretations

The calculations in the previous section indicate that in first order perturbation theory, the usual electric field has the effect of causing the covariant bound state to decay, a phenomenon known from the exact, non-perturbative treatment of the Stark effect. However, the observed shifting of the spectral lines, understood semi-classically as the alignment of the bound state's effective dipole moment in the external electric field, is not reproduced from this contribution. In order to recover the usual Stark splitting, it was necessary to introduce a scalar potential which depends linearly on the position four-vector. This scalar potential has a natural interpretation in the pre-Maxwell electromagnetic theory, which we now present.

Consider the one particle Stueckelberg equation,

$$i\partial_\tau \psi(x, \tau) = \left[ \frac{p_\mu p^\mu}{2M} + V(x) \right] \psi(x, \tau) . \quad (116)$$

Saad, Horwitz, and Arshansky have argued [13] that the local gauge covariance of equation (116) should include transformations which depend on  $\tau$ , as well as on the spacetime coordinates. This requirement of full gauge covariance leads to a theory of five gauge compensation fields, since gauge transformations are functions on the five dimensional space  $(x, \tau)$ . Under local gauge transformations of the form

$$\psi(x, \tau) \rightarrow e^{ie_0\Lambda(x, \tau)} \psi(x, \tau) \quad (117)$$

the equation

$$-(i\partial_\tau - e_0 a_5)\psi(x, \tau) = \frac{1}{2M}(p^\mu - e_0 a^\mu)(p_\mu - e_0 a_\mu)\psi(x, \tau) \quad (118)$$

is covariant, when the compensation fields transform as

$$a_\mu(x, \tau) \rightarrow a_\mu(x, \tau) + \partial_\mu \Lambda(x, \tau) \quad a_5(x, \tau) \rightarrow a_5(x, \tau) + \partial_\tau \Lambda(x, \tau). \quad (119)$$

The Schrödinger-like equation (118) leads to the five dimensional conserved current

$$\partial_\mu j^\mu + \partial_\tau j^5 = 0 \quad (120)$$

where

$$j^5 = |\psi(x, \tau)|^2 \quad j^\mu = \frac{-i}{2M} (\psi^* (\partial^\mu - ie_0 a^\mu) \psi - \psi (\partial^\mu - ie_0 a^\mu) \psi^*). \quad (121)$$

In analogy to nonrelativistic quantum mechanics the squared amplitude of the wave function may be interpreted as the probability of finding an event at  $(\tau, x)$ . Equation (120) may be written as  $\partial_\alpha j^\alpha = 0$ , with  $\alpha = 0, 1, 2, 3, 5$ .

According to (118), we can write the classical Hamiltonian as

$$K = \frac{1}{2M} (p^\mu - e_0 a^\mu) (p_\mu - e_0 a_\mu) - e_0 a_5 \quad (122)$$

and using the Hamilton equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \quad \frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial x_\mu} \quad (123)$$

we find

$$M \dot{x}^\mu = (p^\mu - e_0 a^\mu) \quad (124)$$

which enables us to write the classical Lagrangian,

$$\begin{aligned} L &= \dot{x}^\mu p_\mu - K \\ &= \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + e_0 \dot{x}^\mu a_\mu + e_0 a_5. \end{aligned} \quad (125)$$

We may find the Lorentz force [15] by applying the Euler-Lagrange equations to (125), which in the notation  $\alpha, \beta = 0, 1, 2, 3, 5$ , is

$$M \ddot{x}^\mu = f^\mu{}_\nu \dot{x}^\nu + f^\mu{}_5 = f^\mu{}_\alpha(x, \tau) \dot{x}^\alpha. \quad (126)$$

where

$$f^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu \quad f^\mu{}_5 = \partial^\mu a_5 - \partial_\tau a^\mu. \quad (127)$$

The four equations (126) imply [15]

$$\frac{d}{d\tau} \left( \frac{1}{2} M \dot{x}^2 \right) = M \dot{x}^\mu \ddot{x}_\mu = \dot{x}^\mu (f_{\mu 5} + f_{\mu\nu} \dot{x}^\nu) = \dot{x}^\mu f_{\mu 5} \quad (128)$$

So, the conditions for the dynamical conservation of  $\dot{x}^2 = \text{constant}$ , are

$$f_{5\mu} = 0 \quad \text{and} \quad \partial_\tau f^{\mu\nu} = 0 \quad (129)$$

Thus, the mass-shell relation has the status, classically, of a conservation law (a constant of motion conserved by Noether's theorem for the  $\tau$ -translation symmetry) rather than a constraint.

When we add as the dynamical term for the gauge field,  $(\lambda/4)f_{\alpha\beta}f^{\alpha\beta}$  where  $\lambda$  is a dimensional constant, the equations for the field are found to be

$$\partial_\beta f^{\alpha\beta} = \frac{e_0}{\lambda} j^\alpha = e j^\alpha \quad (130)$$

$$\epsilon^{\alpha\beta\gamma\delta\epsilon} \partial_\alpha f_{\beta\gamma} = 0 \quad (131)$$

where  $f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$ , and

$$j^\mu(\tau, y) = \dot{x}^\mu(\tau) \delta^4(y - x(\tau)) \quad (132)$$

$$j^5(\tau, y) = \rho(\tau, y) = \delta^4(y - x(\tau)) . \quad (133)$$

We identify  $e_0/\lambda$  as the dimensionless Maxwell charge (it follows from (138) below that  $e_0$  has dimension of length). The three vector form of the pre-Maxwell equations are

$$\begin{aligned} \nabla \cdot \mathbf{e} &= e j^0 + \partial_\tau \varepsilon^0 & \nabla \times \mathbf{e} + \partial_0 \mathbf{h} &= 0 \\ \nabla \times \mathbf{h} - \partial_0 \mathbf{e} - \partial_\tau \varepsilon &= e \mathbf{j} & \nabla \cdot \mathbf{h} &= 0 \\ \nabla \cdot \varepsilon &= e j^4 - \partial_0 \varepsilon^0 & \nabla \times \varepsilon - \sigma \partial_\tau \mathbf{h} &= 0 \\ \nabla \varepsilon^0 &= -\sigma \partial_\tau \mathbf{e} - \partial_0 \varepsilon \end{aligned} \quad (134)$$

where

$$\begin{aligned} e_i &= f^{0i} & h_i &= \frac{1}{2} \epsilon_{ijk} f^{jk} \\ \varepsilon^i &= f^{5i} & \varepsilon^0 &= f^{50} \end{aligned} \quad (135)$$

Since the 4-vector part of the current in (121) is not conserved by itself, it may not be the source for the Maxwell field. However, integration of (121) over  $\tau$ , with appropriate boundary conditions, leads to  $\partial_\mu J^\mu = 0$ , where

$$J^\mu(x) = \int_{-\infty}^{\infty} d\tau j^\mu(x, \tau) \quad (136)$$

so that we may identify  $J^\mu$  as the source of the Maxwell field. Under appropriate boundary conditions, integration of (130) over  $\tau$  implies

$$\partial_\nu F^{\mu\nu} = eJ^\mu \quad \epsilon^{\mu\nu\rho\lambda} \partial_\mu F_{\nu\rho} = 0 \quad (137)$$

where

$$F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau f^{\mu\nu}(x, \tau) \quad A^\mu(x) = \int_{-\infty}^{\infty} d\tau a^\mu(x, \tau) \quad (138)$$

so that  $a^\alpha(x, \tau)$  has been called the pre-Maxwell field.

In the pre-Maxwell theory, interactions take place between events in spacetime rather than between worldlines. Each event, occurring at  $\tau$ , induces a current density in spacetime which disperses for large  $\tau$ , and the continuity equation (120) states that these current densities evolve as the event density  $j^5$  progresses through spacetime as a function of  $\tau$ . As noted above, if  $j^5 \rightarrow 0$  as  $|\tau| \rightarrow \infty$  (pointwise in spacetime), then the integral of  $j^\mu$  over  $\tau$  may be identified with the Maxwell current. This integration has been called concatenation [14] and provides the link between the event along a worldline and the notion of a particle, whose support is the entire worldline. Concatenation places the electromagnetic field on the zero mass-shell. The Maxwell theory has the character of an *equilibrium limit* of the microscopic pre-Maxwell theory.

In consideration of the pre-Maxwell theory, the scalar action-at-a-distance potential in the Horwitz-Piron quantum theory, may be seen as an effective interaction resulting from the scalar gauge potential  $a_5$ . This effective interaction follows from the concatenation process, by which microscopic  $\tau$ -dependent evolution is averaged, according to

$$e_0 a_5(x, \tau) \xrightarrow{\text{average}} e_0 \left[ \frac{1}{\lambda} \int d\tau a_5(x, \tau) \right] = eA_5(x) = -V(x) \quad (139)$$

so that the scalar potential plays the role of the Coulomb potential in nonrelativistic mechanics.

If we consider a scalar potential of the form

$$V'(x) = -e A^5(x) = -e \varepsilon^\mu x_\mu \quad (140)$$

with constant  $\varepsilon^\mu$ , then — since  $A^\mu(x)$  is independent of  $\tau$  — the corresponding the field strength tensor will be

$$F^{5\mu} = \partial^5 A^\mu - \partial^\mu A^5 = \varepsilon^\mu . \quad (141)$$

We see from (141) that the choice of scalar potential required to recover the Stark splitting from the covariant bound state theory corresponds precisely to a constant external four-vector electric field  $F^{5\mu} = \varepsilon^\mu$ , analogous to the constant external three-vector electric field  $F^{0j} = E^j$  which causes the bound state to decay. This interpretation of the Stark effect calculation suggests that the parameterized evolution theories of the Stueckelberg type require the pre-Maxwell electromagnetic theory as a corollary, in order to provide a complete description of known phenomenology.

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